

# ON THE ORTHOGONAL BASIS OF THE SYMMETRY CLASSES OF TENSORS ASSOCIATED WITH THE SEMI-DIHEDRAL GROUPS

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**ABSTRACT.** A necessary and sufficient condition for the existence of an orthogonal basis of decomposable symmetrized tensors for the symmetry classes of tensors associated with Semi-Dihedral groups is given. Here the necessary condition is independent of the permutation structures of these groups.

## 1. INTRODUCTION

Let  $V$  be an  $n$ -dimensional complex inner product space and  $G$  be a permutation group on  $m$  elements. Let  $\chi$  be any irreducible character of  $G$ . For any  $\sigma \in G$ , define the operator

$$P_\sigma : \bigotimes_{i=1}^m V \rightarrow \bigotimes_{i=1}^m V$$

by

$$P_\sigma(v_1 \otimes \dots \otimes v_m) = (v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(m)}) \quad (1)$$

The symmetry classes of tensors associated with  $G$  and  $\chi$  is the image of the symmetry operator

$$T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P_\sigma, \quad (2)$$

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and it is denoted by  $V_\chi(G)$ . We say that the tensor  $T(G, \chi)(v_1 \otimes \dots \otimes v_m)$  is a decomposable symmetrized tensor, and we denote it by  $v_1 * \dots * v_m$ . The inner product on  $V$  induces an inner product on  $V_\chi(G)$  which satisfies

$$\langle v_1 * \dots * v_m, u_1 * \dots * u_m \rangle = \frac{\chi(1)}{|G|} d_\chi^G(A),$$

where  $A = [\langle v_j, u_j \rangle]$ , and  $d_\chi^G(A)$  is the generalized matrix function,

$$d_\chi^G(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^m a_{i, \sigma(i)}.$$

Let  $\Gamma_n^m$  be the set of all sequences  $\alpha = (\alpha_1, \dots, \alpha_m)$ , with  $1 \leq \alpha_i \leq n$ . Define the action of  $G$  on  $\Gamma_n^m$  by

$$\sigma.\alpha = (\alpha_{\sigma^{-1}(1)} \otimes \dots \otimes \alpha_{\sigma^{-1}(m)}).$$

Let  $O(\alpha) = \{\sigma.\alpha \mid \sigma \in G\}$  be the *orbit* of  $\alpha$ . We write  $\alpha \sim \beta$  if  $\alpha$  and  $\beta$  belong to the same orbit in  $\Gamma_n^m$ . Let  $\Delta$  be a system of distinct representatives of the orbits. We denote by  $G_\alpha$  the *stabilizer subgroup* of  $\alpha$ , i.e.,  $G_\alpha = \{\sigma \in G \mid \sigma.\alpha = \alpha\}$ . Define

$$\Omega = \{\alpha \in \Gamma_n^m \mid \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0\},$$

and put  $\overline{\Delta} = \Delta \cap \Omega$ .

Let  $\{e_1, \dots, e_n\}$  be an orthonormal basis of  $V$ . Now let us Denote by  $e_\alpha^*$  the tensor  $e_\alpha(1) * \dots * e_\alpha(m)$ . We have

$$\langle e_\alpha^*, e_\beta^* \rangle = \begin{cases} 0 & \text{if } \alpha \neq h.\beta, \\ \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\beta} \chi(\sigma h^{-1}) & \text{if } \alpha = h.\beta. \end{cases}$$

In particular, for  $\sigma_1, \sigma_2 \in G$  and  $\gamma \in \overline{\Delta}$  we obtain

$$\langle e_{\sigma_1.\gamma}^*, e_{\sigma_2.\gamma}^* \rangle = \frac{\chi(1)}{|G|} \sum_{x \in \sigma_2 G_\gamma \sigma_1^{-1}} \chi(x). \quad (3)$$

Moreover,  $e^* \neq 0$  if and only if  $\alpha \in \Omega$ .

For  $\alpha \in \overline{\Delta}$ ,  $V_\alpha^* = \langle e_{\sigma, \gamma}^* : \sigma \in G \rangle$  is called the orbital subspace of  $V_\chi(G)$ . It follows that

$$V_\chi(G) = \bigoplus_{\alpha \in \overline{\Delta}} V_\alpha^*$$

is an orthogonal direct sum. In [8] it is proved that

$$\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma) = \chi(1)[\chi, 1_{G_\alpha}]. \quad (4)$$

Thus we deduce that if  $\chi$  is a linear character, then  $\dim V_\alpha^* = 1$  and in this case the set

$$\{e_\alpha^* | \alpha \in \overline{\Delta}\}$$

is an orthogonal basis of  $V_\chi(G)$ . A basis which consists of the decomposable symmetrized tensors  $e_\alpha^*$  is called an *orthogonal \*-basis*. If  $\chi$  is not linear, it is possible that  $V_\chi(G)$  has no orthogonal \*-basis. The reader can find further information about the symmetry classes of tensors in [1-8] and [10-17]. In this paper a necessary and sufficient condition for the existence of an orthogonal basis of decomposable symmetrized tensors for the symmetry classes of tensors associated with a class of Semi-Dihedral groups is given. Here our method is closed to the technique used in [4] and [16].

## 2. GENERALITIES

**Semi-Dihedral groups** of order  $4n$  ( $n \geq 3$ ) is defined by

$$SD_{4n} := \langle a, b | a^{4n} = b^2 = 1, bab^{-1} = a^{2n-1} \rangle, \quad n \geq 3.$$

**Theorem 2.1.** For each integer  $1 \leq h \leq n-1$ ,  $SD_{4n}$  has a non-linear character  $\chi_h$  of degree 2 which is irreducible given by

$$\chi_h(a^r) = 2 \cos \frac{\pi r h}{n}, \quad \chi_h(a^r b) = 0, \quad 0 \leq r < 4n.$$

where other characters are linear.

**Proof.** Write  $\xi = e^{\frac{\pi i}{n}}$ . For each integer  $j$  with  $1 \leq j < n-1$ , define

$$A_j = \begin{pmatrix} \xi^j & 0 \\ 0 & \xi^{-j} \end{pmatrix}, \quad B_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (5)$$

by using the condition  $n \in \Gamma$ , we have

$$A_j^{4n} = B_j^2 = 1, \quad B_j A_j B_j^{-1} = A_j^{2n-1}.$$

Hence similar to [9, p.183] one can obtain the characters as James and Liebeck found for dihedral groups .

Let  $G := SD_{4n}$  then

**Lemma 1.** Let  $H$  be a subgroup of  $G$ . Then there is a natural number  $r$ ,  $0 \leq r < 4n$  such that  $H = \langle a^r \rangle$  or  $\langle a^r \rangle < H$  and  $H \cap \langle a \rangle = \langle a^r \rangle$ .

**Proof.** One can prove it similar to [4, p.143].

**Lemma 2.** Suppose  $1 \leq h \leq n-1$ ,  $0 \leq r < 4n$ . let  $l = (\frac{2n}{\gcd(2n,r)})$ . Then we have

$$\sum_{t=1}^l \cos(\frac{trh\pi}{n}) = \begin{cases} l, & \text{if } rh \equiv 0 \pmod{2n} \\ 0, & \text{if } rh \not\equiv 0 \pmod{2n} \end{cases}$$

**Proof.** It is straightforward.

**Lemma 3.** Suppose  $\chi = \chi_h$ ,  $1 \leq h \leq n-1$ . let  $H$  be any subgroup of  $G$ . If  $l = (\frac{2n}{\gcd(2n,r)})$ , then we have

$$\sum_{g \in H} \chi(g) = \begin{cases} 2l, & \text{if } rh \equiv 0 \pmod{2n} \\ 0, & \text{if } rh \not\equiv 0 \pmod{2n} \end{cases}$$

**Proof.** By using lemma 1,  $H \cap \langle a \rangle = \langle a^r \rangle$  for some natural number  $r$ ,  $0 \leq r < 4n$ . Base of theorem1,  $\chi$  vanishes outside  $\langle a \rangle$ , therefore by lemma 2 we have

$$\sum_{g \in H} \chi(g) = \sum_{t=1}^l \chi(a^{tr}) = 2 \sum_{t=1}^l \cos(\frac{trh\pi}{n}) = \begin{cases} 2l, & rh \equiv 0 \pmod{2n} \\ 0, & rh \not\equiv 0 \pmod{2n} \end{cases}$$

**Lemma 4.** Let  $\chi = \chi_h$ ,  $1 \leq h \leq n-1$ . Then for  $\gamma \in \overline{\Delta}$ , we have  $G_\gamma = \langle a^r \rangle$  or  $\langle a^r \rangle < G_\gamma$  and  $G_\gamma \cap \langle a \rangle = \langle a^r \rangle$ , for some  $r$ ,  $0 \leq r < 4n$ , where  $rh \equiv 0 \pmod{2n}$ .

**Proof.** Since  $G_\gamma$  is a subgroup of  $G$  by lemma 1,  $G_\gamma = \langle a^r \rangle$  or  $\langle a^r \rangle < G_\gamma$  and  $G_\gamma \cap \langle a \rangle = \langle a^r \rangle$ , for some  $r$ ,  $0 \leq r < 4n$ . But by lemma

3 if  $rh \not\equiv 0 \pmod{2n}$ , then  $\sum_{g \in G_\gamma} \chi(g) = 0$  which shows  $\gamma \notin \overline{\Delta}$ . Thus  $rh \equiv 0 \pmod{2n}$ .

**Recall.** If  $1 \leq h \leq n-1$  and  $h = m_1 m_2$  with  $m_1$  a power of 2 and  $m_2$  odd, then  $\nu_2(\frac{h}{n}) < 0$  if and only if  $\nu_2(\frac{m_1 m_2}{n}) < 0$  if and only if  $2m_1 | n$ .

**Lemma 5.** Let  $1 \leq h \leq n-1$ . Then there exist  $t, t', 0 \leq t, t' < 4n$ , such that  $\cos(\frac{(t-t')h\pi}{n}) = 0$  if and only if  $\nu_2(\frac{h}{n}) < 0$ , where  $\nu_2$  is the 2-adic valuation.

**Proof.** It is straightforward.

**Theorem 2.2.** Let  $G = SD_{2k}$  be a subgroup of  $S_m$ , let  $\chi = \chi_h$ ,  $1 \leq h \leq n-1$  and assume  $d = \dim V \geq 2$ , Then  $V_\chi(G)$  has an orthogonal \*-basis if and only if  $\nu_2(\frac{h}{n}) < 0$ .

**Proof.** It is enough to prove that for any  $\alpha \in \overline{\Delta}$  the orbital subspace  $V_\alpha^*$  has orthogonal \*-basis if and only if  $\nu_2(\frac{h}{n}) < 0$ . Let  $\nu_2(\frac{h}{n}) < 0$  and assume  $\alpha \in \overline{\Delta}$ . By Lemma 4,  $G_\alpha = \langle a^r \rangle$  or  $\langle a^r \rangle < G_\alpha$ . Let  $o(a^r) = (\frac{2n}{\gcd(2n, r)})$ . Now we consider two cases.

**Case 1.** If  $G_\alpha = \langle a^r \rangle = \{a^r, a^{2r}, \dots, a^{lr} = 1\}$ , then by (4) and lemma 3,

$$\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma) = \frac{2}{l}(2l) = 4.$$

For any  $\sigma_1, \sigma_2 \in G$ , we have

$$\sigma_2 G_\alpha \sigma_1^{-1} = \begin{cases} \{a^{r+i-j}, a^{2r+i-j}, \dots, a^{lr+i-j}\}, & \text{if } \sigma_1 = a^j, \sigma_2 = a^i \\ \{a^{r+i+j(1-2n)}b, a^{2r+i+j(1-2n)}b, \dots, a^{lr+i+j(1-2n)}b\}, & \text{if } \sigma_1 = a^j b, \sigma_2 = a^i \\ \{a^{(1-2n)i-r-j}, a^{i(1-2n)-2r-j}, \dots, a^{i(1-2n)-lr-j}\}, & \text{if } \sigma_1 = a^j b, \sigma_2 = a^i b \end{cases}$$

If  $\sigma_1 = a^j, \sigma_2 = a^i$ , by (3) we have

$$\begin{aligned} \langle e_{\sigma_1 \cdot \alpha}^*, e_{\sigma_2 \cdot \alpha}^* \rangle &= \frac{\chi(1)}{|G|} \sum_{x \in \sigma_2 G_\alpha \sigma_1^{-1}} \chi(x) = \frac{2}{2n} \sum_{t=1}^l \chi(a^{tr+i-j}) \\ &= \frac{2}{n} \sum_{t=1}^l \cos \frac{(tr+i-j)h\pi}{n} = \frac{2}{n} \sum_{t=1}^l \cos \left( \frac{trh\pi}{n} + \frac{(i-j)h\pi}{n} \right) \end{aligned}$$

$$= \frac{2}{n} \sum_{t=1}^l \cos\left(\frac{(i-j)h\pi}{n}\right) = \frac{2l}{n} \cos\left(\frac{(i-j)h\pi}{n}\right) \quad (6)$$

Where the equality of the one before last is due to lemma 4.  
If  $\sigma_1 = a^j b, \sigma_2 = a^i$ , we have

$$\langle e_{\sigma_1.\alpha}^*, e_{\sigma_2.\alpha}^* \rangle = 0$$

and for  $\sigma_1 = a^j b, \sigma_2 = a^i b$ , we have

$$\begin{aligned} \langle e_{\sigma_1.\alpha}^*, e_{\sigma_2.\alpha}^* \rangle &= \frac{\chi(1)}{|G|} \sum_{x \in \sigma_2 G_\gamma \sigma_1^{-1}} \chi(x) = \frac{2}{2n} \sum_{t=1}^l \chi(a^{i(1-2n)-tr-j}) \\ &= \frac{2}{n} \sum_{t=1}^l \cos \frac{(i(1-2n) - tr - j)h\pi}{n} \\ &= \frac{2}{n} \sum_{t=1}^l \cos\left(\frac{-trh\pi}{n} + \frac{(i-j)h\pi}{n} - \frac{2nh\pi}{n}\right) \\ &= \frac{2}{n} \sum_{t=1}^l \cos\left(\frac{(i-j)h\pi}{n}\right) = \frac{2l}{n} \cos\left(\frac{(i-j)h\pi}{n}\right) \quad (7) \end{aligned}$$

Where the equality of the one before last is due to lemma 4.

Therefore

$$\langle e_{\sigma_1.\alpha}^*, e_{\sigma_2.\alpha}^* \rangle = \begin{cases} \frac{2l}{n} \cos\left(\frac{(i-j)h\pi}{n}\right), & \text{if } \sigma_1 = a^j, \sigma_2 = a^i \\ 0, & \text{if } \sigma_1 = a^j b, \sigma_2 = a^i \\ \frac{2l}{n} \cos\left(\frac{(i-j)h\pi}{n}\right), & \text{if } \sigma_1 = a^j b, \sigma_2 = a^i b \end{cases}$$

Since  $\nu_2(\frac{h}{n}) < 0$ , hence by Lemma 5 there exist  $t_1, t_2, 0 \leq t_1, t_2 < 4n$  such that  $\cos(\frac{(t_1-t_2)h\pi}{n}) = 0$ . Put

$$S = \{a^{t_1}.\alpha, a^{t_2}.\alpha, a^{t_1}b.\alpha, a^{t_2}b.\alpha\} \subseteq \Gamma_n^m.$$

Then for every  $\gamma, \beta \in S$  and  $\gamma \neq \beta$  we have

$$\langle e_\alpha^*, e_\beta^* \rangle = 0$$

But  $\dim V_\alpha^* = 4$ ; hence  $\{e_\xi^* | \xi \in S\}$  is an orthogonal \*-basis for  $V_\alpha^*$ .

**Case 2.** If  $\langle a^r \rangle < G_\alpha$ , then

$$\langle a^r \rangle = \langle a \rangle \cap G_\alpha = \{a^r, a^{2r}, \dots, a^{lr} = 1\}$$

also  $|G_\alpha| \geq 2l$ . Thus by (4),

$$\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma) \leq \frac{2}{2l}(2l) = 2.$$

Therefore  $\dim V_\alpha^* = 1$  or  $\dim V_\alpha^* = 2$ .

If  $\dim V_\alpha^* = 1$ , then it is obvious that we have an orthogonal  $*$ -basis.

Suppose  $\dim V_\alpha^* = 2$ , then by Lemma 5 there exist  $i, j, 0 \leq i, j < 4n$  such that  $\cos(\frac{(i-j)h\pi}{n}) = 0$ . Set  $\sigma_1 = a^j, \sigma_2 = a^i$ . Then

$$\sigma_2 G_\alpha \sigma_1^{-1} \cap \langle a \rangle = \{a^{r+i-j}, \dots, a^{lr+i-j}\}.$$

Hence by (3) and (6) we have

$$\langle e_{a^j, \alpha}^*, e_{a^i, \alpha}^* \rangle = 0$$

Which means  $\{e_{\sigma_1, \alpha}^*, e_{\sigma_2, \alpha}^*\}$  is an orthogonal  $*$ -basis for  $V_\alpha^*$ .

**Conversely**, let  $V_\alpha^*$  has an orthogonal  $*$ -basis for every  $\alpha \in \overline{\Delta}$ . Now we prove that  $\nu_2(\frac{h}{n}) < 0$ .

Let  $a = a_1 \dots a_r$  be the disjoint cycle decomposition of  $a$  in  $S_m$  and for every  $1 \leq i \leq r$ , let  $o(a^i) = m_i$ . Since  $o(a) = n$ , so  $n = [m_1, \dots, m_r]$ , the least common multiple of  $m_1, \dots, m_r$ . Set  $a_i = (i_1, i_2, \dots, i_{m_i})$  and define  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \Gamma_n^m$  in such a way that  $\alpha_{i_1} = 1$  and  $\alpha_{i_2} = \alpha_{i_3} = \dots = \alpha_{i_{m_i}} = 2$ , for every  $1 \leq i \leq r$ , and for  $j \in \{1, 2, \dots, m\} \setminus \bigcup_{i=1}^r \{i_1, \dots, i_{m_i}\}$ ,  $\alpha_j = 1$ . Now similar to [16, p.642], one can see that  $\nu_2(\frac{h}{n}) < 0$ .

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